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THE FRONT END OF A GIVEN VOLUME HAVING OPTIMUM PRESSURE DRAG IN THE APPROXIMATION OF NEWTON'S LAW OF RESISTANCE*

A.N. KRAIKO

The problem of determining the profile of the front end of a two-dimensional or axisymmetric body of given volume having minimum pressure drag, as predicted by Newton's law of resistance, is considered. In this formulation, the solution depends on the magnitude of the non-dimensional volume $\omega = \Omega/(\pi^\nu Y^{2+\nu})$, where $\nu = 0$ or 1 in the two-dimensional and axisymmetric cases, respectively, Y is the maximum admissible radius or half-thickness (if $\nu = 0$) of the body, which will henceforth be taken as the unit of length, and Ω is the given volume (or half of the latter in the case $\nu = 0$) of the desired forebody.

The range of optimum shape calculations that have been based on Newton's law of resistance, beginning with Newton's own efforts, is so large (see /1, 2/) that yet another excursion into that area in the same approximation might seem unjustified, all the more so, as the case $\nu = 1$ has already been treated /3/. However, that attempt was far from successful. Indeed, according to /3/, the solution for the finite ω range

$$0.149 \simeq 313\sqrt{3}/3645 < \omega < \sqrt{3}/5 \simeq 0.346$$

"if it exists, cannot be determined under the present formulation of the problem". For $0 \leqslant \omega \leqslant 313\sqrt{3}/3645$ the optimum front end has a blunt nose; while for $\sqrt{3}/5 \leqslant \omega \leqslant \infty$ it must have a cusp /3/. It was mainly the lack of any solution in the intermediate range that prompted us to revise the treatment given in /3/. The result of this revision was, first, a solution valid for any $\omega \leqslant \infty$. Second, we were able not only to show that all the results obtained in /3/ were incorrect and to construct optimal front ends for $\omega \leqslant \infty$, but also to show that for $0 \leqslant \omega \leqslant 13/30 \simeq 0.433$ the optimum shape is essentially new: a cusp projecting from a blunt nose.

1. Consider a two-dimensional or axisymmetric front end af attached at $x = 0$ to a cylindrical body (Fig.1,a) whose radius or half-thickness, as already mentioned, is taken as the unit of length, i.e., $y_f = 1$. Throughout, x, y are Cartesian or cylindrical coordinates, according as $\nu = 0$ or $\nu = 1$; the x axis is directed along the velocity vector $\mathbf{v}_\infty$ of the free stream and either coincides with the axis or lies in the plane of symmetry of the body; indices $a, \ldots, f$ will be attached to the parameters at the appropriate points of the contour af; the index ∞ will indicate the free-stream parameters. The pressure p at the body surface is given by Newton's formula:

$$p = p_\infty + \rho_\infty v_\infty^2 \sin^2\vartheta \tag{1.1}$$

Here ρ is the density, $v = |\mathbf{v}|$, and ϑ is the angle of inclination of the contour with respect to the x axis. Formula (1.1) holds for $0 \leqslant \vartheta \leqslant \pi/2$ or, if $y = y(x)$ or $x = x(y)$ is the equation of af, for

$$0 \leqslant y^{\cdot} \equiv dy/dx \leqslant \infty,\ 0 \leqslant x' \equiv dx/dy \leqslant \infty. \tag{1.2}$$

We wish to construct a generator af which, for the given volume ω, while satisfying (1.2) and the size constraint

$$y \leqslant y_f = 1 \tag{1.3}$$

minimizes the pressure drag χ. Under these conditions the quantity χ itself is determined by the distribution of p over af, which is found from $y^{\cdot}$ or $x' \equiv 1/y^{\cdot}$ at each point of af, using (1.1).

In this formulation the most general shape of af is a collection of bilateral extremum segments on which $0 < y^{\cdot} < \infty$ or $0 < x' < \infty$, plus vertical $(x' \equiv 0)$ and horizontal $(y \equiv 1$ or $y^{\cdot} \equiv 0)$ limiting extremum segments, which appear because of the constraints (1.2) and (1.3). This is shown schematically in Fig.1,b; any of the segments drawn in the figure, e.g., the "flat nose" (FN) ab, the "inner vertical segment" (IVS) du or the "end cylinder" (EC) hf,

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may not be present. There may be more than one flat inner face.

It follows from (1.1) that, apart from a constant term (in χ) and constant factors for configurations of the type shown in Fig.1,b, which are not essential for the variational problem as formulated here, we have

$$\chi = \int_{<} \varphi(y, y^{\cdot})\, dx + \int_{\perp} x'\varphi\left(y, \frac{1}{x'}\right) dy \tag{1.4}$$

$$\omega = \int_{<} \Phi(y)\, dx + \int_{\perp} x'\Phi(y)\, dy$$

$$\varphi(y, y^{\cdot}) = \frac{y^{\nu} y^{\cdot 3}}{1 + y^{\cdot 2}}, \quad \Phi(y) = y^{1+\nu}$$

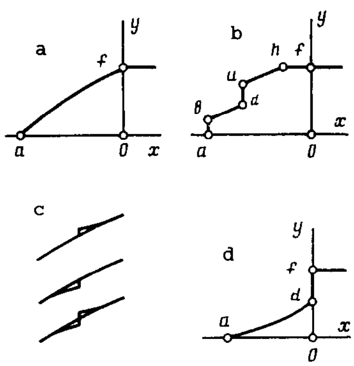

Fig.1

Here the sign $<$ indicates the part of the integral along af over segments with $x' > 0$; the sign $\perp$ has the same meaning for flat sections.

To obtain the necessary conditions for a minimum of χ, which determine the optimum generator af, using a constant Lagrange multiplier λ, one usually constructs an auxiliary functional $I = \chi + \lambda\omega$, whose admissible variations δI, with ω maintained constant, are identical with $\delta\chi$. Therefore, considering χ and ω as in (1.4), and varying an arbitrary - i.e., not necessarily optimum - contour af containing the segments shown in Fig.1,b we obtain

$$\begin{aligned}
\delta\chi \equiv \delta I = &\left\{\left[\frac{2y^{\nu}y^{\cdot 3}}{(1+y^{\cdot 2})^2} - \lambda y^{1+\nu}\right]_{b+} - \left[\frac{2y^{\nu}y^{\cdot 3}}{(1+y^{\cdot 2})^2}\right]_{a}\right\}\Delta x_a + \\
&\left[y^{\nu}\frac{1-y^{\cdot 2}}{(1+y^{\cdot 2})^2}\right]_{b+}\Delta y_b + \left\{\left[\lambda y^{1+\nu} - \frac{2y^{\nu}y^{\cdot 3}}{(1+y^{\cdot 2})^2}\right]_{d-} + \right. \\
&\left.\left[\frac{2y^{\nu}y^{\cdot 3}}{(1+y^{\cdot 2})^2} - \lambda y^{1+\nu}\right]_{u+}\right\}\Delta x_d + \left[y^{\nu}\frac{y^{\cdot 2}-1}{(1+y^{\cdot 2})^2}\right]_{d-}\Delta y_d + \\
&\left[y^{\nu}\frac{1-y^{\cdot 2}}{(1+y^{\cdot 2})^2}\right]_{u+}\Delta y_u + \left[y^{\nu}\frac{y^{\cdot 2}(3+y^{\cdot 2})}{(1+y^{\cdot 2})^2}\right]_{h-}\Delta y_h - \\
&\left[\frac{2y^{\nu}y^{\cdot 3}}{(1+y^{\cdot 2})^2}\right]_{h-}\Delta x_h + \int_{\perp}\left\{\lambda\left(y^{1+\nu} - y_{b,u}^{1+\nu}\right) + \right. \\
&\left.\left[\frac{2y^{\nu}y_{+}^{\cdot 3}}{(1+y_{+}^{\cdot 2})^2}\right]_{b,u}\right\}\delta x' dy - \lambda(1+\nu)\int_{\dashv}(x - x_h)\,\delta y^{\cdot}\, dx + \\
&\int_{<}\left(\Phi_y - \frac{d}{dx}\Phi_{y^{\cdot}}\right)\delta y\, dx, \quad \Phi(y, y^{\cdot}) = \varphi(y, y^{\cdot}) + \lambda\Phi(y)
\end{aligned} \tag{1.5}$$

Here Δx and Δy are the increments of the coordinates, in the general case, of the corner points, where segments of different types meet; the subscripts "minus" ("plus") indicate values "before", i.e., to the left of ("after", i.e., to the right of) the corners; $\delta x'$ is the variation-increment of x' for fixed y on flat sections, and δy and $\delta y^{\cdot}$ are the variations-increments of y and $y^{\cdot}$ for fixed x on other segments; the term with subscript "a" in the coefficient of Δx_a will appear only if there is no FN (otherwise it vanishes automatically, since then $x_a' = 0$ but $y_a^{\cdot} = \infty$), but in that case there are no terms in the expression for $\delta\chi$ with subscript "b"; the subscript "b, u" means that the subscript of the appropriate upper point (either b or u) is retained in the integral over the FN or over an IVS; if there are several inner flats, the terms with subscript "d" and "u" are summed over all of them; finally, the symbol "$\dashv$" signifies an integral over an end cylinder (if it exists).

Without going into the details of the well-known technique by which formula (1.5) is obtained, we will just note two equalities used for that purpose:

$$\Delta x_{b,u} = \Delta x_{a,d} + \int_{\perp}\delta x' dy, \quad \int_{\dashv}\delta y\, dx = \int_{\dashv}(x_h - x)\,\delta y^{\cdot}\, dx \tag{1.6}$$

The integrals in (1.5) and (1.6), as well as in (1.4), are evaluated in the direction of increasing x and y.

2. Analysis of formula (1.5) yields the necessary conditions for χ to be a minimum, which determine the optimum contour af. We begin with the fact that, since λ is chosen at will, there must be a "compensating" point k in one of the segments with $0 < y^{\cdot} < \infty$ at which the coefficient of δy in the last integral vanishes, i.e., we put

$$\Phi_y - \frac{d}{dx}\Phi_{y'} \equiv 2y^{\nu}\left[\lambda\,\frac{1+\nu}{2} - \frac{\nu y'^3}{y(1+y'^2)^2} - y'y''\,\frac{3-y'^2}{(1+y'^2)^3}\right] = 0 \qquad (2.1)$$

After this the other variations (δy outside a small neighbourhood of k on the "inclined" sections of $af, \delta x'$ on the FN and IVSs and $\delta y'$ on the EC hf) and increments $\Delta x_a, \Delta y_b, \ldots$ may be considered to be independent. This independence is achieved by stipulating that when any of these quantities are varied simultaneously, y varies in the neighbourhood of k but in such a way that the volume ω remains constant. At the same time, however, since Eq.(2.1) holds at k, the contribution to $\delta\chi$ due to the non-zero value of δy near k in the last integral of (1.5) is a quantity of higher order of smallness. Since the variations δy on segments with $0 < y' < \infty$ are arbitrary, it follows that Eq.(2.1) is true not only at k but on all segments which are bilateral extremum segments (BES). Since Φ is independent of x, Eq.(2.1), as we know, is integrable and gives

$$\Phi - y'\Phi_{y'} \equiv y^{\nu}\left[\lambda y - \frac{2y'^3}{(1+y'^2)^2}\right] = C \qquad (2.2)$$

for a suitably determined constant C.

If the integrand in (1.5) includes quantities of the order of $(\delta y')^2$ over BESs, this gives what is known as Legendre's necessary condition for φ as in (1.4) and arbitrary $\Phi(y)$: $y' \leqslant \sqrt{3}$. However, this inequality, which makes the coefficient of $(\delta y')^2$ non-negative, is too weak; it must be replaced - again for any function $\Phi(y)$ - by

$$y' \leqslant 1 \qquad (2.3)$$

This constraint follows at once from an examination of the coefficients of Δy_d or Δy_u in (1.5), since if it is not true then the insertion of an infinitesimal IVS into a BES, such that $\Delta y_d \leqslant 0$ but $\Delta y_u \geqslant 0$, will reduce χ (see Fig.1,c).

Now suppose that the optimal generator includes a vertical segment (a flat nose or inner vertical segment). Then Δy_b, Δy_d and Δy_u are arbitrary, and therefore

$$y_{\pm}' = 1 \qquad (2.4)$$

i.e., the BES approaches any vertical segment at an angle of 45°. Similarly, as Δy_h is arbitrary,

$$y_{h-}' = 0 \qquad (2.5)$$

so that, if the optimum forebody (OF) contains an EC, the BES must approach it smoothly. On the EC itself it is permitted that $\delta y' \geqslant 0$. Therefore the optimality condition (non-negativity of $\delta\chi$) would be

$$\lambda \leqslant 0 \qquad (2.6)$$

On the other hand, if the OB contains a flat nose, it follows from the arbitrariness of Δx_a, from the fact that $x_a' = 0$ and from (2.4) that

$$y_b^{\nu}\left[\frac{2y'^3}{(1+y'^2)^2} - \lambda y\right]_{b+} \equiv y_b^{1+\nu}\left(\frac{1}{2y_b} - \lambda\right) = 0 \qquad (2.7)$$

Consequently, in that case $\lambda = 1/(2y_b)$, which is inconsistent with (2.6). Therefore the OF cannot have both a FN and an EC. Similarly, assuming the presence of an IVS, noting that Δx_d is arbitrary and using Eq.(2.4), we obtain

$$\lambda = \frac{y_u^{\nu} - y_d^{\nu}}{2(y_u^{1+\nu} - y_d^{1+\nu})} \qquad (2.8)$$

It can be shown that the λ of (2.7) does not equal the λ of (2.8), implying that the OF cannot have both a FN and IVSs.

We will now show that the OF for a given volume does not contain a FN. To that end, isolating y'' from (2.1) and using (2.4) and (2.7), we find that $y_{b+}'' = 1/y_b > 0$. Thus y' must increase as one moves right from the point b_+. Since $y' = 1$, this violates condition (2.3). Consequently, the desired generator does not contain a FN. For a cusped OF, we equate the coefficient of Δx_a in (1.5) to zero, getting

$$y_a' = 0 \qquad (2.9)$$

Hence it follows that the constant C in (2.2) must vanish for at least the "first" BES ad (there is now no point b!), so its Eq.(2.2) becomes

$$\lambda y - \frac{2y'^3}{(1+y'^2)^2} = 0 \qquad (2.10)$$

Now suppose that the OF includes an IVS. Then it follows from Eq.(2.10), which is valid at d_-, and from the condition obtained by equating the coefficient of Δx_d in (1.5) to zero (see above, the derivation of (2.8)), that (2.10) remains valid to the right of the IVS, thus also at the point u_+. By (2.4), however, this gives different values of the constant λ, so we have proved that there cannot be IVSs. Finally, it follows from (2.10) and (2.5) that the OF cannot contain an EC.

According to the above analysis, therefore, the desired OF contains neither FN, IVSs nor EC, and if it has a cusped BES the latter must satisfy condition (2.3): $y^\cdot \leqslant 1$, which is more restrictive than Legendre's condition $y^\cdot \leqslant \sqrt{3}$ used in /3/. Consequently, the flat-nosed front ends and sharp-nosed contours, with $y^\cdot > 1$ at the ends, constructed in /3/, are not optimal. This would seem to make the problem even more acute, since now we lack an OF over an even larger range of ω values than indicated in /3/.

In actual fact, however, our analysis has overlooked one more possibility for the position of a vertical section: at $x = 0$, i.e., an OF consisting of an "end vertical section" (EVS) with a cusp projecting from it (Fig.1,d). A formula for $\delta\chi$ in this case is obtained by omitting the terms in (1.5) subscripted b and h, as well as the integral over the horizontal segment, and setting $y_{u+}^\cdot = 0$, $y_{u+} = y_f = 1$. Then, using the arbitrariness of the variations δy over ad and of Δy_d, we again obtain Eq.(2.10), which determines the shape of the optimum cusp ad, and condition (2.4): $y_{d-}^\cdot = 1$ at its end point. The remaining terms in (1.5), involving Δx_d and $\delta x'$ over df, may be rewritten in a more convenient form after integrating the integral over the EVS by parts:

$$\delta\chi = -\frac{y_d^{\nu}}{2}\Delta x_d - \lambda(1+\nu)\int\limits_{\perp} y^\nu \delta x\, dy$$

Since by (2.10) $\lambda > 0$, this implies that under any admissible variation of the EVS, when $\Delta x_d \leqslant 0$ and $\delta x \leqslant 0$ on df, we have $\delta\chi \geqslant 0$ and therefore this configuration is indeed optimal.

3. The solution of Eq.(2.10) can be developed in parametric form. To that end, taking $q \equiv y^\cdot$ as parameter and remembering that by (2.4) and (2.10) $\lambda = 1/(2y_d)$, we can rewrite (2.10) in the form

$$y^\circ \equiv \frac{y}{y_d} = y^\circ(q) \equiv \frac{4q^3}{(1+q^2)^2} \tag{3.1}$$

Hence, putting $x^\circ \equiv x/y_d$, we have

$$\frac{dx^\circ}{dq} = \frac{dx^\circ}{dy^\circ}\frac{dy^\circ}{dq} = \frac{1}{q}\frac{d}{dq}\left[\frac{4q^3}{(1+q^2)^2}\right]$$

Integrating from d, where $q = 1$ and $x^\circ = 0$, to an arbitrary non-negative $q < 1$, we obtain

$$x^\circ = x^\circ(q) \equiv 2\frac{q^2-1}{(1+q^2)^2} \tag{3.2}$$

Eqs.(3.1) and (3.2), in which $0 \leqslant q \leqslant 1$ (see (2.4) and (2.9)), are the required parametric representation of the optimum cusp. By (3.2), $x_a^\circ \equiv x^\circ(0) = -2$, i.e., the length of the peak is $2y_d$. Eqs.(3.1) and (3.2) involve neither ν nor y_d. Consequently, the shape of the optimum cusp is the same in the two-dimensional and axisymmetric cases and, in addition, it is universal in the variables x° and y° (independent of the value of $y_d \leqslant 1$). The function $y^\circ = y^\circ(x^\circ)$ as constructed by formulae (3.1) and (3.2) is shown in Fig.2.

By (1.4), (3.1) and (3.2), the expressions for ω and χ for the OF consisting of a vertical section and a cusp projecting from it are as follows:

$$\omega = \begin{cases} \left(\frac{1}{3}+\frac{\pi}{8}\right) y_d^2 & \text{if } \nu = 0 \\ (13/30)\, y_d^3 & \text{if } \nu = 1 \end{cases} \tag{3.3}$$

$$\chi = \begin{cases} 1-\left(\frac{1}{3}+\frac{\pi}{8}\right) y_d = 1 - \sqrt{\left(\frac{1}{3}+\frac{\pi}{8}\right)\omega} & \text{if } \nu = 0 \\ \frac{1}{2} - \frac{13}{40} y_d^2 = \frac{1}{2} - \frac{13}{40}\left(\frac{30}{13}\omega\right)^{2/3} & \text{if } \nu = 1 \end{cases}$$

The second equalities for χ are obtained by replacing y_d by ω according to the formulae for ω.

Formulae (3.3) hold as long as y_d does not reach its "limiting" value $y_d = 1$, i.e., provided that

$$0 \leqslant \omega \leqslant 1/3 + \pi/8 \simeq 0.726 \text{ if } \nu = 0 \qquad (3.4)$$
$$0 \leqslant \omega \leqslant 13/30 \simeq 0.433 \text{ if } \nu = 1$$

Incidentally, it should be mentioned that for front ends with straight-line generators (wedges or cones) and half-angle $\alpha = 45°$ at the cusp, $\omega = 1/[2(1+\nu)]$, i.e., $\omega = 0.5$ or 0.25, according as $\nu = 0$ or $\nu = 1$. For arbitrary α

$$\omega = \frac{1}{(2+\nu)\operatorname{tg}\alpha}, \quad \chi = \frac{\operatorname{tg}^2\alpha}{(1+\nu)(1+\operatorname{tg}^2\alpha)} = \frac{1}{(1+\nu)[1+(2+\nu)^2\omega^2]} \qquad (3.5)$$

where $\operatorname{tg}\alpha$ may take values in the range $0 \leqslant \operatorname{tg}\alpha \leqslant \infty$. The limiting cases are $\alpha = \pi/2$, $\operatorname{tg}\alpha = \infty$, $\omega = 0$ and $\chi = 1/(1+\nu)$, corresponding to a flat nose, and $\alpha = \operatorname{tg}\alpha = \chi = 0$ and $\omega = \infty$, corresponding to an infinitely long cusp with straight-line generator.

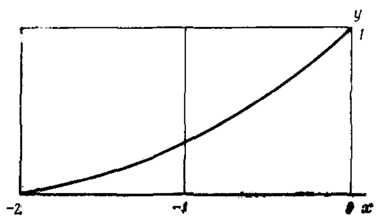

Fig.2

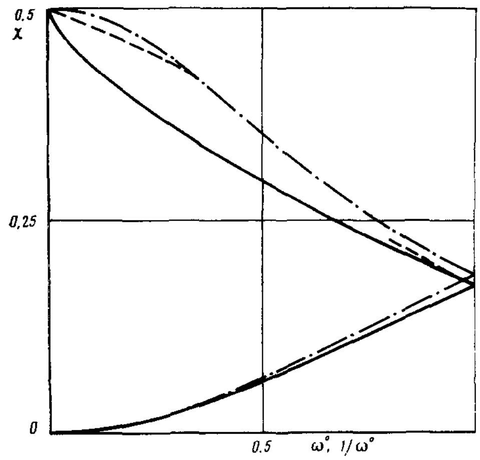

Fig.3

For ω greater than the limiting values of (3.4), the optimum case is that of cusped bodies whose contours, beginning on the x axis, where $q_a \equiv \dot{y}_a = 0$, reach the point f with $x_f = 0,\ y_f = 1$ and $q_f < 1$. A parametric representation of these bodies is obtained along the same lines as (3.1) and (3.2):

$$y = \frac{2q^3}{\lambda(1+q^2)^2}, \quad x = \frac{1}{\lambda}\left[\frac{q^2-1}{(1+q^2)^2} - \frac{q_f^2-1}{(1+q_f^2)^2}\right] \qquad (3.6)$$
$$\lambda = 2q_f^3/(1+q_f^2)^2, \quad 0 \leqslant q \leqslant q_f < 1$$

Naturally, if $q_f = 1$ and $y_d = 1$ these formulae are identical with (3.1) and (3.2). Moreover, it can be shown that the OFs (3.6) are "scaled" terminal parts of the cusp illustrated in Fig.2. Allowing for the differences in the choice of variables, formulae (3.6) are equivalent to those obtained for cusped front ends in /3/. The essential difference, however, is that in /3/ it was claimed that such front ends are optimal in the larger range $q_f \leqslant \sqrt{3}$. The integral characteristics ω and χ of these bodies are given by

$$\omega = \frac{45 + 6q_f^2 + q_f^4}{120 q_f}, \quad \chi = \frac{15 - q_f^2}{40(1+q_f^2)} q_f^2, \quad 0 \leqslant q_f \leqslant 1 \qquad (3.7)$$

- these formulae are identical with those derived in /3/ (except for the notation and the limiting value of q_f).

The curves $\chi = \chi(\omega)$ and $\chi = \chi(\omega^{-1})$ of the axisymmetric OFs constructed above, of the allegedly "optimum" front ends with $\nu = 1$ in /3/ and of the cones given by (3.3), (3.5) and (3.6), are shown in Fig.3. In order to cover the entire range of ω values, two curves have been drawn for each case: the upper curves $\chi = \chi(\omega)$ for $0 \leqslant \omega \leqslant 13/30$ and the lower curves through the origin, $\chi = \chi(\omega^{-1})$ for $13/30 \leqslant \omega \leqslant \infty$. The solid curves represent the OFs worked out in the present paper, the dashed curves those of /3/ (broken off in the range of ω values for which no solution was obtained in /3/) and, finally, the dash-dot curves represent cones. The horizontal axis represents both $\omega° = \omega/(13/30)$ and $(\omega°)^{-1}$. At $\omega° > 1$, i.e., $(\omega°)^{-1} < 1$, the lower solid and dashed curves coincide.

It should be stressed that our solutions have been constructed in the approximation of

Newton's law of resistance which, as is well-known /4/, works much better for convex bodies than for concave ones. One can expect particularly large deviations from this theory in bodies with "inner positive" corners, such as the corner at the point d in configurations with rear vertical segments. The first way to guarantee a better approximation to reality is to introduce an additional restriction on the radius of curvature of the admissible contours: $R \geqslant r$, where $r > 0$ is some given constant. When that is done the corner at d is rounded off to radius r. Another method would be to introduce point forces at "positive" corners. The question of modifying the solution in the context of this approach requires additional anaysis.

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SELFSIMILAR SOLUTIONS DESCRIBING THERMAL CAPILLARY FLOWS IN VISCOUS LAYERS*

V.A. BATYSHCHEV

Thermal capillary flows in thin layers, brought about by non-uniform heating of the free boundary, are investigated at high Marangoni numbers. Selfsimilar solutions of the non-linear boundary-layer equations are constructed under conditions of axial symmetry, and asymptotic formulae for the solutions are found for small and large values of the thickness of the layer. It is shown that the selfsimilar solutions may not be unique when the parameters of the problem have certain values. The buoyancy forces in an inhomogeneous fluid lead to the reinforcement or suppression of the flows or to the formation of reverse flows close to the free boundary. Selfsimilar solutions when there are thermal capillary effects present have been studied in /1-5/.

1. The non-linear axially-symmetric problem of the stationary thermal capillary motion of an incompressible fluid in a thin layer, bounded by a free surface Γ and a solid wall S is considered at low coefficients of viscosity $\nu \to 0$ and thermal diffusivity $\chi \to 0$ when there is a zero temperature gradient on the free boundary:

$$(\mathbf{v}, \nabla)\,\mathbf{v} = -\rho^{-1}\nabla p + \nu\Delta\mathbf{v} + \mathbf{g} \tag{1.1}$$
$$\mathbf{v}\nabla T = \chi\Delta T, \ \operatorname{div}\mathbf{v} = 0$$

$$p = 2\nu\rho\mathbf{n}\Pi\mathbf{n} + \sigma(k_1 + k_2) + p_*, \ (x, y, z) \in \Gamma \tag{1.2}$$
$$2\nu\rho\,[\Pi\mathbf{n} - (\mathbf{n}\Pi\mathbf{n})] = \nabla_\Gamma\sigma, \ \mathbf{v}\mathbf{n} = 0, \ (x, y, z) \in \Gamma$$
$$T = T_\Gamma, \ (x, y, z) \in \Gamma; \ \mathbf{v} = T - T_S = 0, \ (x, y, z) \in S$$

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